ISRAEL JOURNAL OF MATHEMATICS 95 (1996), 135–155

GROUPS IN THE CLASS SEMIGROUPS OF VALUATION DOMAINS

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ABSTRACT

It is shown that the isomorphy classes of the ideals of a valuation domain form a Clifford semigroup, and the structure of this semigroup is investigated. The group constituents of this Clifford semigroup are exactly the quotients of totally ordered complete abelian groups, modulo dense subgroups. A characterization of these groups is obtained, and some realization results are proved when the skeleton of the totally ordered group is given.

Introduction

Let R be a commutative domain, with quotient field $Q \neq R$. The set $\mathcal{F}(R)$ of the non-zero fractional ideals of R is a commutative semigroup under multiplication, with R as unit. $\mathcal{F}(R)$ contains the subgroups $\mathcal{I}(R) \geq \mathcal{P}(R)$ consisting, respectively, of the invertible fractional ideals and of the non-zero principal fractional ideals.

The factor group $\mathcal{C}(R) = \mathcal{I}(R)/\mathcal{P}(R)$ is the class group of R; it can be viewed as the group of the isomorphy classes of the invertible ideals of R, with multiplication induced by multiplication of ideals. If R is a Dedekind domain, it is one of the main objects of investigation in algebraic number theory; in this case $\mathcal{I}(R) = \mathcal{F}(R)$. Groups related to domains R, obtained by generalizing the class group $\mathcal{C}(R)$ in various ways, have been investigated.

* The authors are members of GNSAGA of CNR. This research was supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica, Italy.
Received April 15, 1993 and in revised form September 28, 1994

The factor semigroup $\mathcal{S}(R) = \mathcal{F}(R)/\mathcal{P}(R)$ can be viewed as the semigroup of the isomorphy classes of non-zero (integral) ideals of R, with multiplication induced by multiplication of ideals: $\mathcal{S}(R) = \{[I]\colon 0 < I \leq R\}, [I][J] = [IJ]$. We call $\mathcal{S}(R)$ the **class semigroup** of R.

The class semigroup $\mathcal{S}(R)$ has not received as much attention as the class group even for simple classes of domains, probably because the semigroup structure is much less attractive than the group structure. However, there are some types of commutative semigroups whose structure is so simple and so close to the group structure that it is natural to look for domains R whose class semigroup $\mathcal{S}(R)$ is of that type. For example, semigroups which are close to groups are the Clifford semigroups: these are disjoint unions of groups.

The first goal of this paper is to investigate the class semigroup of a valuation domain R. After a preliminary section where we collect basic results on commutative semigroups and ideals of valuation domains, we will prove, in the second section, that the class semigroup $\mathcal{S}(R)$ of a valuation domain R is a Clifford semigroup satisfying particular properties, and provide some examples of class semigroups of valuation domains.

It is worth remarking that the class semigroup $\mathcal{S}(R)$ of the valuation domain R is determined by the value group $\Gamma(R)$, and gives much less information on the ring R than $\Gamma(R)$ itself. As a matter of fact, two valuation domains with very different value groups — even with different prime spectra — can have isomorphic class semigroups, as we shall see by some examples. Thus our paper mainly addresses the structure of a totally ordered abelian group Γ; one can define the semigroup $\mathcal{S}(\Gamma)$ associated with Γ as the set of equivalence classes of proper filters in Γ (called **classes supérieures** by Ribenboim [R]), with respect to the equivalence relation $\approx$ defined by setting: $F_1 \approx F_2$ if $F_1 = \gamma + F_2$ for some $\gamma \in \Gamma$ (F_i filters in Γ). All the results on the class semigroup $\mathcal{S}(R)$ can be phrased in terms of the semigroup $\mathcal{S}(\Gamma)$.

In the third section, we achieve the second aim of the paper: a characterization of abelian groups that can be group constituents of $\mathcal{S}(R)$. They are associated with the idempotent non-zero prime ideals L of R; the group associated with such an ideal L, denoted by G_L, is isomorphic to $\overline{\Gamma}_L/\Gamma_L$ (where Γ_L is the value group of the localization R_L of R at the prime ideal L) and $\overline{\Gamma}_L$ is its completion in the order topology. First, a *coarse characterization* of the groups G_L, which disregards the rest of the structure of Γ, is given; then some realization results for these groups are obtained when the skeleton of Γ is fixed. Classical results on algebraically compact and cotorsion abelian groups play a fundamental role in this section, as well as Nöbeling's celebrated solution of the Specker problem [N] and its generalization given by Kaup and Keane [KK].

The characterization of the groups G_L obtained here is also interesting in view of the results in [BFS]; these groups appear in that paper as building blocks of the Clifford semigroup Unis_L^K of the isomorphy classes of the uniserial R-modules U of type $[K/I]$ and $U_\# = I^\# = L$ (for these notions and notation we refer to the paper [BFS]).

Finally, we would like to mention that Zanardo and Zannier [ZZ] investigate the Clifford semigroup $\mathcal{S}(R)$ for certain domains R, in particular for orders in algebraic number rings. The results in the present paper and in Zanardo-Zannier's paper raise the question of characterizing Prüfer domains R such that $\mathcal{S}(R)$ is a Clifford semigroup.

We are indebted to Adalberto Orsatti and Paolo Zanardo for helpful discussions.

1. Preliminaries

For notions and results of classical abelian group theory used later on we refer to the books by Fuchs [F2] and Orsatti [O]. We will just recall here some basic facts concerning commutative semigroups and ideals of valuation domains.

1.1 Commutative Semigroups. Let $\mathcal{S}$ be a commutative (multiplicative) semigroup with 1. Consider the subsemigroup of the idempotent elements

$$\mathrm{Id}(\mathcal{S}) = \{e \in \mathcal{S}\colon e = e^2\}.$$

It is well known that $\mathrm{Id}(\mathcal{S})$ is a Λ-semilattice under multiplication. Consider now the subsemigroup of $\mathcal{S}$ consisting of the von Neumann regular elements

$$\mathrm{Reg}(\mathcal{S}) = \{a \in \mathcal{S}\colon a = axa, \exists x \in S\};$$

$\mathrm{Reg}(\mathcal{S})$ is a Clifford semigroup, since it is the disjoint union of the family of groups $\{G_e\}_{e \in \mathrm{Id}(\mathcal{S})}$ indexed by the idempotents, where

$$G_e = \{ae\colon abe = e, \exists b \in \mathcal{S}\},$$

and it is the maximal group in $\mathcal{S}$ containing the idempotent e. If $e \leq f$ are idempotents (i.e. $ef = e$), then the multiplication by e induces a group homomorphism $\phi_e^f\colon G_f \to G_e$, which is called the **bonding homomorphism** between G_f and G_e. Notice that G_1 is the subgroup of $\mathcal{S}$ consisting of the invertible elements. Obviously $\mathcal{S}$ is a Clifford semigroup if and only if $\mathcal{S} = \mathrm{Reg}(\mathcal{S})$.

If one considers the equivalence relation $\sim$ on $\mathcal{S}$ defined by setting: $a \sim b$ iff $\mathcal{S}a = \mathcal{S}b$, then all elements which are equivalent to an element $a \in \mathrm{Reg}(\mathcal{S})$ are contained in $\mathrm{Reg}(\mathcal{S})$, and there exists an idempotent e (namely $e = ax$, if $a = axa$) such that the equivalence class of a coincides with G_e.

1.2 Ideals of Valuation Domains. Let R be a valuation domain with quotient field $Q \neq R$ and maximal ideal P. Given a fractional ideal I of R, let $I^{-1} = \{q \in Q\colon qI \leq R\}$, and let $I^{\#} = \{r \in R; rI < I\} = \bigcup\{qI\colon q \in Q, qI < R\}$ be the prime ideal associated with I (see [FS, page 15]). It is well known that $I^{\#} = P$ if $I \cong R$, and $I^{\#} = II^{-1}$ if $I \ncong R$; moreover, I is a fractional ideal of $R_{I^{\#}}$, the localization of R at $I^{\#}$, hence $IR_{I^{\#}} = I$. If $I \cong R_L$ for some prime ideal L, then necessarily $L = I^{\#}$.

Lemma 1: *Let I be a fractional ideal of R. The following hold:*
(1) *if $I \cong R_{I^{\#}}$, then $II^{-1} = R_{I^{\#}}$ and $II^{\#} \cong I^{\#}$;*
(2) *if $I \ncong R_{I^{\#}}$, then $II^{-1} = I^{\#}$ and $II^{\#} = I$.*

Proof: See Lemma 1.1 in [BFS]. ∎

The groups in $\mathcal{S}(R)$ associated with the idempotent classes $[R_L]$, for $0 \neq L$ a prime ideal, and $[L]$, for $0 \neq L = L^2$ an idempotent prime ideal, will be simply denoted respectively by G_{R_L} and G_L. This notation is justified by the fact that two different prime ideals are not isomorphic, neither are their localizations.

The next result gives a characterization of G_L; recall that $L = L^2$ is equivalent to $L \ncong R_L$ (see [FS, I.4.8]).

Lemma 2: *Let $0 \neq L = L^2$ be an idempotent prime ideal of R. Then the subgroup G_L of $\mathcal{S}(R)$ consists of the isomorphy classes $[I]$ of ideals I such that $I^{\#} = L$ and $I \ncong R_L$.*

Proof: $[I] \in G_L$ iff $I \cong HL$ for an ideal H and there exists a fractional ideal J such that $IJL = L$; since $(IJL)^{\#} = I^{\#} \cap J^{\#} \cap L^{\#}$ by [BFS, Lemma 1.2], and $L^{\#} = L$ by [FS, I.4.5], the equality $IJL = L$ implies that $I^{\#} \geq L$. Since

$I^{\#} = H^{\#} \cap L$, it follows that $I^{\#} = L$. $I \cong R_L$ implies $HL \cong R_L$ hence $L \cong R_L$, which contradicts $L = L^2$, therefore $I \ncong R_L$.

Conversely, if $I^{\#} = L$ and $I \ncong R_L$, then $II^{-1} = L$, by Lemma 1, hence $IJL = L$ for $J = I^{-1}$, so that $[I] \in G_L$. ∎

The group G_L is defined in [BFS] by means of the characterization given in Lemma 2.

2. Class semigroups of valuation domains

In this section R always denotes a valuation domain. The main goal is to prove the following theorem concerning the structure of the class semigroup $\mathcal{S}(R)$. Note that, if $L > L'$ are two prime ideals, then in $\mathcal{S}(R)$ we have: $[R_L] \geq [L] > [R_{L'}] \geq [L']$.

THEOREM 3: *Let R be a valuation domain. Then:*

1. *the Λ-semilattice $\mathrm{Id}(\mathcal{S}(R))$ of the idempotents of $\mathcal{S}(R)$ consists of the isomorphy classes $[R_L]$ of the localizations of R at non-zero prime ideals L and of the isomorphy classes $[L]$ of the idempotent non-zero prime ideals $L = L^2$; moreover, $\mathrm{Id}(\mathcal{S}(R))$ is totally ordered;*
2. *the groups G_{R_L} associated with the idempotent $[R_L]$ is trivial, for all non-zero prime ideals L; if $L = L^2$ is non-zero, the group G_L associated with the idempotent $[L]$ is isomorphic to $\overline{\Gamma}_L/\Gamma_L$, where Γ_L is the value group of R_L and $\overline{\Gamma}_L$ is its completion in the order topology;*
3. *$\mathcal{S}(R)$ is a Clifford semigroup and the bonding homomorphisms between its group constituents are all trivial.*

Proof: (1) If either $[I] = [R_L]$ for a prime ideal L, or $[I] = [L]$ for an idempotent prime ideal $L = L^2$, then $[I]$ is obviously idempotent. Conversely, let $[I] = [I^2]$ be an idempotent in $\mathcal{S}(R)$, where $I \leq R$. Then there exists an $a \in R$ such that $aI = I^2$. If $I \cong R_L$ (in which case necessarily $L = I^{\#}$), then we are done. If $I \ncong R_{I^{\#}}$, then, by Lemma 1, $aI^{\#} = aII^{-1} = I^2I^{-1} = II^{\#} = I$, hence $[I] = [L]$ for $L = I^{\#}$. To conclude, it is enough to recall that $L \cong R_L$ if and only if $L > L^2$. Concerning the total order on $\mathrm{Id}(\mathcal{S}(R))$, note that $[R_L] \geq [L]$ is either a strict inequality or an equality depending on the fact that $L = L^2$ or $L > L^2$; $L \ncong R_{L'}$ and $LR_{L'} = R_{L'}$.

(2) Recall that

$$G_{R_L} = \{[IR_L]\colon IJR_L = R_L, \exists J \in \mathcal{F}(R)\}.$$

$IJR_L = R_L$ obviously implies $IR_L \cong R_L$, hence the first claim holds. The proof that, if $L = L^2$, then G_L is isomorphic to $\overline{\Gamma}_L/\Gamma_L$, is in [BFS, Proposition 6.3] (where the characterization of G_L given in Lemma 2 is used).

(3) If $[I] = [I]^2$, then $[I]$ is trivially a regular element of $\mathcal{S}(R)$. If $[I] \neq [I]^2$, then $I \not\cong R_{I^\#}$, therefore $II^{-1} = I^\#$, hence $I^2I^{-1} = II^\# = I$, by Lemma 1, so $[I] = [I][I^{-1}][I]$ is regular. Therefore $\mathcal{S}(R)$ is a Clifford semigroup. The bonding homomorphisms from G_{R_L} are obviously trivial, since G_{R_L} itself is trivial. Given two prime ideals $L > L'$, the bonding homomorphism from G_L to $G_{R_{L'}}$ is also trivial, since $G_{R_{L'}}$ is trivial. ∎

Recall that $\Gamma_L = \Gamma(R_L)$ is isomorphic to Γ/Σ, where $\Gamma = \Gamma(R)$ is the value group of R and Σ is the convex subgroup associated with the prime ideal L of R.

In order not to exclude the prime ideal $\{0\}$ from our consideration, we can adjoin to the class semigroup $\mathcal{S}(R)$ two more elements, namely $[\{0\}]$ and $[Q] = [R_{\{0\}}]$; the multiplication is extended in the obvious way: if $[I] \in \mathcal{S}(R)$, then

$$[\{0\}][I] = [\{0\}]; \quad [Q][I] = [Q]; \quad [\{0\}][Q] = [\{0\}].$$

The two new elements are idempotents with trivial associated groups $G_{\{0\}}$ and G_Q. The **extended class semigroup** $\mathcal{S}^*(R) = \mathcal{S}(R) \cup [\{0\}] \cup [Q]$ is still a Clifford semigroup, with $[I] > [Q] > [\{0\}]$ for all $[I] \in \mathcal{S}(R)$.

In the rest of this section we will describe some properties of the class semigroup of a valuation domain.

COROLLARY 4: *The extended class semigroup $\mathcal{S}^*(R)$ of a valuation domain R has the following properties:*

(1) *if the maximal subgroup G_e containing the idempotent e is non-trivial, then e has an immediate successor e^+ and G_{e^+} is trivial;*
(2) *$\mathrm{Id}(\mathcal{S}^*(R))$ is order complete;*
(3) *if an idempotent is the supremum of strictly smaller idempotents, then it has an immediate successor e^+ and G_{e^+} is trivial.*

Proof: (1) Theorem 3 shows that G_e is non-trivial only if $e = [L]$, where $L = L^2$ is an idempotent non-zero prime ideal of R. In this case e has $e^+ = [R_L]$ as an immediate successor, and G_{R_L} is trivial;

(2) the intersection and the union of prime ideals are prime ideals, so the supremum and the infimum of idempotents associated with a family of prime

ideals $\{L_i\}_{i\in I}$ coincide respectively with $[R_L]$, where $L=\bigcap L_i$, and $[L']$, where $L'=\bigcup L_i$;

(3) if $e=\sup_{i\in I} e_i$, with $e>e_i$ for all i, then e is necessarily of the form $[L]$, for L an idempotent prime ideal, union of strictly smaller prime ideals, hence e has $e^+=[R_L]$ as immediate successor. ∎

The following result describes the structure of the groups G_L under certain hypotheses.

PROPOSITION 5: *Let L be an idempotent prime ideal of a valuation domain R. Then:*

(1) *if there exists a maximum prime ideal L' properly contained in L, then G_L is isomorphic to $\mathbb{R}/\Delta$, the factor group of the additive group $\mathbb{R}$ of the reals modulo a dense subgroup Δ;*

(2) *if L is the countable union of smaller prime ideals, then G_L is a cotorsion group.*

Proof: (1) The proof is already sketched in [BFS]: owing to (2) in Theorem 3, Γ_L has a minimal non-zero convex subgroup isomorphic to a dense subgroup Δ of the additive group of the reals, and $\overline{\Gamma}_L/\Gamma_L$ is canonically isomorphic to $\overline{\Delta}/\Delta=\mathbb{R}/\Delta$ (one could also use the isomorphism $\overline{\Gamma}_L\cong(\Gamma_L/\Delta)\oplus\mathbb{R}$, proved in [R, page 27], which however uses the incorrect isomorphism $\Gamma_L\cong(\Gamma_L/\Delta)\oplus\Delta$).

(2) In view of the correspondence between prime ideals of R_L and convex subgroups of Γ_L (see [FS, I.3]), there is a countable descending chain of convex (hence pure) subgroups of Γ_L

$$\Gamma_L=\Gamma_0\supseteq\Gamma_1\supseteq\Gamma_2\supseteq\cdots\supseteq\Gamma_n\supseteq\cdots$$

such that $\bigcap\{\Gamma_n\colon n\in\omega\}=\{0\}$. In [DO] it is shown that the map

$$\phi\colon \prod_{n\in\omega}\Gamma_n\to\overline{\Gamma}_L$$

defined by setting: $\phi\big((\gamma_n)_{n\in\omega}\big)=\Sigma_{n\in\omega}\gamma_n$ ($\gamma_n\in\Gamma_n$), induces an epimorphism from $\prod_{n\in\omega}\Gamma_n/\bigoplus_{n\in\omega}\Gamma_n$ onto the group $\overline{\Gamma}_L/\Gamma_L$; hence $\overline{\Gamma}_L/\Gamma_L$ is cotorsion, since it is a quotient of the algebraically compact group $\prod_{n\in\omega}\Gamma_n/\bigoplus_{n\in\omega}\Gamma_n$ (see [H]). ∎

It is not difficult to construct an example of a valuation domain R containing a non-zero idempotent prime ideal L such that G_L is not algebraically compact:

see the first example in the following series of examples of class semigroups of valuation domains. We will use the following notation: if A is a totally ordered abelian group and κ is a cardinal number, then A^κ denotes the direct product totally ordered by the lexicographic order, and $A^{<\kappa}$ denotes its subgroup consisting of the elements with support of cardinality strictly smaller than κ.

Examples: (1) Let $\mathbb{Z}^{\omega_1}$ be totally ordered by the lexicographic order; $\mathbb{Z}^{\omega_1}$ is complete in the order topology and $\mathbb{Z}^{<\omega_1}$ is dense in it. The quotient $\mathbb{Z}^{\omega_1}/\mathbb{Z}^{<\omega_1}$ is $\aleph_1$-free (i.e. every countable subgroup is free, see [EM, Exercise 11, page 113]), cotorsionfree (see [D]), and it has no free summands, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\omega_1}/\mathbb{Z}^{<\omega_1}, \mathbb{Z}) = 0$ by [L]. If R is a valuation domain with $\Gamma(R) = \mathbb{Z}^{<\omega_1}$ and P as maximal ideal, then G_P is not cotorsion.

(2) A valuation domain R is discrete of rank one (rank=Krull dimension) if and only if $\mathcal{S}(R)$ is the trivial group.

(3) Let R be a rank one non-discrete valuation domain. Then $\mathcal{S}(R)$ consists of two groups: G_R, which is trivial, and G_P, which is trivial if and only if $\Gamma(R)$ is order isomorphic to $\mathbb{R}$. In the case $\Gamma(R) \ncong \mathbb{R}$, then G_P is isomorphic to $\mathbb{R}/\Delta$, for Δ a dense subgroup of $\mathbb{R}$, by Proposition 5. In particular, since any subgroup of $\mathbb{R}$ can be realized as $\Gamma(R)$ for some R, G_P can be any divisible group of cardinality at most $2^{\aleph_0}$.

(4) Let R be a valuation domain of finite Krull dimension equal to n. If R is discrete, namely if $\Gamma(R)$ is isomorphic to a direct sum of n copies of $\mathbb{Z}$ under the lexicographic order (equivalently, no non-zero prime ideal is idempotent), then $\mathcal{S}(R)$ consists of n trivial groups, associated with the localizations of R at the non-zero prime ideals. If R is not discrete, then $\mathcal{S}(R)$ contains also the groups G_L corresponding to the idempotent non-zero prime ideals L; G_L is trivial or isomorphic to a proper quotient of $\mathbb{R}$, depending on the completeness or incompleteness of $\Gamma(R_L)$.

(5) The Clifford semigroup $\mathcal{S}$ consisting of $2n$ $(n \geq 1)$ trivial groups with the total order is isomorphic to the class semigroup both of a discrete valuation domain of rank $2n$, and of a valuation domain of rank n such that every prime ideal L is idempotent and $\Gamma(R_L)$ is complete.

(6) Let Γ be an indecomposable abelian group, which is an extension of $\mathbf{Z}_p$ by $\mathbf{Q}$ (see for instance [K, Theorem 19]). Let Γ be totally ordered in such a way that $\mathbf{Z}_p$ is a convex subgroup. Then Γ gives rise to the same Clifford semigroup as $\Gamma' = \mathbf{Z}_p \bigoplus \mathbf{Q}$ lexicographically ordered. There are no embeddings (even not

order preserving) from Γ to Γ'; this example shows that the generalization of the Hahn embedding theorem to "regular" totally ordered abelian groups given in [R, Theorem 3] is not correct.

(7) Let R be a totally branched discrete valuation domain, i.e. a domain such that $\mathrm{Spec}R$ is well ordered by the opposite inclusion and every non-zero prime ideal is not idempotent. $\mathcal{S}(R)$ is the well ordered union of trivial groups, indexed by $\mathrm{Spec}R\setminus\{0\}$.

3. The group constituents of the Clifford semigroup $\mathcal{S}(R)$

A natural problem arising in our investigation is which abelian groups can be group constituents G_L of the Clifford semigroup $\mathcal{S}(R)$ of a valuation domain R, for some prime ideal L of R. Actually, this is a problem on totally ordered abelian groups; in fact, it can be formulated as follows: find those abelian groups G which are isomorphic to $\overline{(\Gamma/\Sigma)}/(\Gamma/\Sigma)$, where Σ is a convex subgroup of Γ and $\overline{(\Gamma/\Sigma)}$ is the completion of the factor group Γ/Σ in the induced order topology. Obviously, there is no loss of generality in assuming $\Sigma = 0$, which amounts to localizing the valuation domain at the prime ideal L.

Proposition 5 and Example 3 in the preceding section give a satisfactory answer in case Γ has a minimal non-zero convex subgroup: a group G is isomorphic to $\overline{\Gamma}/\Gamma$ for some totally ordered abelian group Γ with a minimal non-zero convex subgroup if and only if G is divisible of cardinality $\leq 2^{\aleph_0}$; hence, hereafter, we will always assume that Γ has no minimal non-zero convex subgroup.

Our goal of characterizing abelian groups G which are isomorphic to $\overline{\Gamma}/\Gamma$ for some totally ordered abelian group Γ without a minimal non-zero convex subgroup, can be achieved in two different ways. The first one is a **coarse characterization**, which considers only the cofinality of the ordered set of the non-zero convex subgroups of Γ; the coarse characterization is obtained in Propositions 7 and 8 below. The other way is by a **fine characterization**, in which the skeleton of Γ is preassigned; we will give a fine characterization for particular skeletons only (see Corollary 13), but we are able to obtain various **fine realization** results (see Theorems 11 and 12).

3.1 THE COARSE CHARACTERIZATION. We deal first with the coarse characterization. Our problem reduces to one on torsionfree abelian groups, in view of the following result essentially due to Simbireva and Neumann (see [F1, Theorem

6, page 48]); recall that a convex subgroup of a totally ordered abelian group is necessarily pure.

THEOREM: *Let Γ be a torsionfree abelian group, κ a cardinal and*

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_\sigma \supseteq \cdots \quad (\sigma < \kappa)$$

a well ordered chain of pure subgroups such that $\bigcap_{\sigma<\lambda} \Gamma_\sigma = \Gamma_\lambda$ for all limit ordinals $\lambda < \kappa$, and $\bigcap_{\sigma<\kappa} \Gamma_\sigma = 0$. Then there exists a total order on Γ such that the subgroups Γ_σ are convex subgroups with respect to this order.

At this point it is useful to introduce the following definition. Given an infinite cardinal κ, we say that the abelian group G is **κ-realizable** if there exists a torsionfree abelian group Γ with a chain of pure subgroups

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_\sigma \supseteq \cdots \quad (\sigma < \kappa)$$

such that $G \cong \overline{\Gamma}/\Gamma$, where $\overline{\Gamma}$ denotes the completion of Γ with respect to the topology which has the given chain as a basis of neighborhoods of zero. The following result, which is almost trivial, is fundamental for our purposes.

LEMMA 6: *The class of κ-realizable abelian groups is closed under quotients.*

Proof: Assume $G \cong \overline{\Gamma}/\Gamma$ is κ-realizable and let $H \cong \Gamma'/\Gamma$ be a subgroup of G, with Γ' a subgroup of $\overline{\Gamma}$ containing Γ. Then $G/H \cong \overline{\Gamma}/\Gamma'$, and $\overline{\Gamma}$ is also the completion of its dense subgroup Γ' with respect to the induced topology; hence, in view of [O, Proposition 3, page 105], G/H is κ-realizable. ∎

In view of Proposition 5 and Example 3 in the preceding section, we must expect different behaviors in the two cases of $\aleph_0$-realizable groups and κ-realizable groups, for κ an uncountable regular cardinal.

The next result gives a sufficient condition for κ-realizability, which is also necessary in the countable case.

PROPOSITION 7: *A cotorsion group is κ-realizable for each infinite regular cardinal κ. An abelian group is $\aleph_0$-realizable if and only if it is cotorsion.*

Proof: If G is cotorsion and reduced then, denoting by D its divisible hull, G is a quotient of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D/G)$, which is a torsionfree algebraically compact group (see [F2, 54.1 and 46.1]); hence, by the preceding lemma, it is enough to show that a torsionfree algebraically compact group H is κ-realizable. Consider any total order on H. There is a pure embedding via the diagonal map of H into $H^\kappa/H^{<\kappa}$. H^κ is complete in the order topology and $H^{<\kappa}$ is dense in H^κ, hence $H^\kappa/H^{<\kappa}$ is κ-representable; since H is pure injective, it is a summand of $H^\kappa/H^{<\kappa}$, hence it is also κ-representable, by Lemma 6. The rest of the claim follows from Proposition 5. ∎

Now we deal with the coarse characterization in the uncountable case.

PROPOSITION 8: *Let κ be an uncountable regular cardinal. Then the class of κ-realizable abelian groups coincides with the class of all abelian groups.*

Proof: Since all groups are quotients of free groups, it is enough to show, by Lemma 6, that a free group F is κ-realizable. Consider any order on F and let F^κ be the lexicographically ordered direct product of κ copies of F; let us think of the elements of F^κ as functions from κ into F. F^κ is complete with respect to the order topology. For every interval I in κ and $f \in F$, let f_{χ_I} be the function assuming the value f for each element of I and 0 elsewhere. Let Γ be the subgroup of F^κ generated by the functions f_{χ_I}, for every $f \in F$ and every interval I of cardinality smaller than κ. It is easy to check that the closure $\overline{\Gamma}$ of Γ in F^κ coincides with the subgroup of F^κ generated by the functions f_{χ_I}, where f ranges in F and I is an arbitrary interval in κ. Clearly $\overline{\Gamma} = \Gamma \oplus F'$, where F' consists of the constant functions. Obviously $F' \cong F$ via the diagonal map, thus $F \cong \overline{\Gamma}/\Gamma$ is κ-representable. ∎

Note that both Γ and $\overline{\Gamma}$ in the preceding proof are Specker groups over F, in the sense defined in [F2, XIII.97]. Clearly every $\aleph_0$-realizable abelian group is κ-realizable, for $\kappa > \aleph_0$, and every group $\mathbb{R}/\Delta$, with Δ dense in $\mathbb{R}$, is also $\aleph_0$-realizable.

3.2 FINE REALIZATIONS. We are going to recall some basic notions on totally ordered abelian groups (see [F1] or [R]). If Γ is such a group, a convex subgroup Σ of Γ is **principal** if it has an immediate predecessor under inclusion. Let $\mathcal{P} = \mathcal{P}(\Gamma)$ denote the set of the principal convex subgroups of Γ with 0 adjoined, ordered by the reverse inclusion. The set $\mathcal{C}$ of all the convex subgroups of Γ is exactly the closure of $\mathcal{P}$ with respect to infima and suprema, hence $\mathcal{P}$ is characterized by the property that its closure is a weakly atomic totally ordered set closed under infima and suprema (see [F1, pages 50–53]). It is worth remarking that, in order to define a **natural valuation** on Γ (see [F1, page 55]), the total ordering considered on $\mathcal{P}(\Gamma)$ is the opposite of the one given by the inclusion, and that it agrees with the ordering on $\mathrm{Id}(\mathcal{S}(\Gamma))$.

We denote by Σ^+ the immediate successor of $\Sigma \in \mathcal{P}$ with respect to this ordering (hence Σ^+ is covered by Σ under the inclusion). For each $\Sigma \in \mathcal{P}$ we denote by $\Delta(\Sigma)$ the factor group Σ/Σ^+; note that $\Delta(\Sigma)$ is order isomorphic to an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$, hence it has infinite cardinality at most $2^{\aleph_0}$. We define $\Delta(0) = 0$.

A skeleton is a system $\mathcal{S} = [\mathcal{P}; \Delta(\Sigma), \Sigma \in \mathcal{P}]$ with $\mathcal{P}$ a totally ordered set with maximum (denoted by 0) whose closure with respect to infima and suprema is a weakly atomic set closed under infima and suprema and with $\Delta(\Sigma)$ totally ordered group isomorphic to an additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$ for each Σ. The skeleton of the totally ordered group Γ is the system $\mathcal{S}(\Gamma) = [\mathcal{P}(\Gamma); \Delta(\Sigma), \Sigma \in \mathcal{P}(\Gamma)]$. In this section we will always assume that $\mathcal{P}(\Gamma)$ has no maximal non-zero elements, i.e. Γ has no minimal non-zero convex subgroup.

Given a skeleton $\mathcal{S}$, we say that the abelian group G is $\mathcal{S}$-**realizable** if it is isomorphic to $\overline{\Gamma}/\Gamma$, where Γ is a totally ordered group with skeleton $\mathcal{S}$.

Given a skeleton $\mathcal{S} = [\mathcal{P}; \Delta(\Sigma), \Sigma \in \mathcal{P}]$, the Hahn product on $\mathcal{S}$: $\mathbb{H}(\mathcal{S}) = \mathbb{H}_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$, is the subgroup of the direct product $\prod_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ consisting of the elements with well ordered support. It is well known that $\mathbb{H}(\mathcal{S})$ is totally ordered by the lexicographic ordering, and its own skeleton coincides with $\mathcal{S}$. Every subgroup of $\mathbb{H}(\mathcal{S})$ containing the direct sum $\bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ has also skeleton $\mathcal{S}$ (with respect to the induced total ordering). Moreover, it is well known that the quotient of the Hahn product modulo every convex subgroup with no immediate successor (under inclusion) is complete in the order topology (see [R, Lemma 9]). Hence every quotient of $\mathbb{H}(\mathcal{S})$ modulo a dense subgroup containing the direct sum $\bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ is $\mathcal{S}$-realizable.

However, these types of $\mathcal{S}$-realizable groups have in general a special structure; for instance, these groups reflect divisibility properties of the groups $\Delta(\Sigma)$. Moreover, if $\mathrm{cof}(\mathcal{P}) = \aleph_0$ ($\mathrm{cof}(\mathcal{P})$ is the regular cardinal given by the cofinality of $\mathcal{P}$), we cannot hope to realize every group, as Proposition 7 indicates. On the other hand, if $\mathrm{cof}(\mathcal{P}) > \aleph_0$, Proposition 8 indicates that we can hope to realize a large class of groups. *Our strategy is to look at totally ordered groups Γ satisfying the inclusions*

$$\text{(a)} \qquad \bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma) \subseteq \Gamma \subseteq \mathbb{H}_{\Sigma\in\mathcal{P}} \Delta(\Sigma);$$

then $\overline{\Gamma}/\Gamma$ and all its quotients are $\mathcal{S}$-realizable. In each case we would like to obtain $\mathcal{S}$-realizable groups of cardinality as large as possible.

It is useful at this point to introduce two cardinal invariants of the skeleton $\mathcal{S}$. We will call the cardinal numbers

$$\phi(\mathcal{S}) = \min\{|\mathbb{H}_{\Sigma\geq\Lambda}\Delta(\Sigma)|: 0 > \Lambda \in \mathcal{P}\} \quad \text{and} \quad \phi(\mathcal{P}) = \min\{|\mathcal{P}(\Lambda)|: 0 > \Lambda \in \mathcal{P}\}$$

final cardinal of $\mathcal{S}$ and **final cardinal of** $\mathcal{P}$, respectively, where $\mathcal{P}(\Lambda) = \{\Sigma \in \mathcal{P}\colon \Lambda \leq \Sigma < 0\}$ is the final interval starting at Λ. The final cardinals are connected with the cofinality of $\mathcal{P}$ and the cardinality of $\mathcal{S}$-realizable groups.

LEMMA 9: *If $\mathcal{S} = [\mathcal{P}; \Delta(\Sigma), \Sigma \in \mathcal{P}]$ is a skeleton, where $\mathcal{P}$ has no maximal non-zero element, and Γ is a totally ordered abelian group with skeleton $\mathcal{S}$, then*
(1) $2^{\mathrm{cof}(\mathcal{P})} \leq \phi(\mathcal{S}) \leq 2^{\phi(\mathcal{P})}$,
(2) $|\overline{\Gamma}/\Gamma| \leq \phi(\mathcal{S})$.

Proof: (1) If $\Lambda \in \mathcal{P}$ is sufficiently close to 0, we have

$$|\mathbb{H}_{\Sigma\geq\Lambda}\Delta(\Sigma)| \leq \Big|\prod_{\Sigma\geq\Lambda}\Delta(\Sigma)\Big| \leq (2^{\aleph_0})^{\phi(\mathcal{P})} = 2^{\phi(\mathcal{P})};$$

on the other hand, for each $0 > \Lambda \in \mathcal{P}$, $\mathcal{P}(\Lambda)$ contains a subset order isomorphic to $\mathrm{cof}(\mathcal{P})$, so that $\mathbb{H}_{\Sigma\geq\Lambda}\Delta(\Sigma)$ contains a subgroup isomorphic to $\prod_{\mathrm{cof}(\mathcal{P})}\mathbb{Z}$, which has cardinality $2^{\mathrm{cof}(\mathcal{P})}$.

(2) Since $\overline{\Gamma}/\Gamma \cong \overline{\Pi}/\Pi$ for every non-zero convex subgroup Π, it is enough to show that, for some Π, $|\overline{\Pi}| \leq \phi(\mathcal{S})$. There is an embedding of Γ into $\mathbb{H}(\mathcal{S}^*)$, with $\mathcal{S}^* = [\mathcal{P}; \Delta(\Sigma)^*, \Sigma \in \mathcal{P}]$, where $\Delta(\Sigma)^*$ is the divisible hull of $\Delta(\Sigma)$ (see [F1]); now $\overline{\Pi}$ coincides with $\overline{\Gamma} \cap \mathbb{H}_{\Sigma\geq\Pi}\Delta(\Sigma)^*$ and, for Π sufficiently close to $0, |\mathbb{H}_{\Sigma\geq\Pi}\Delta(\Sigma)^*| = \phi(\mathcal{S}^*)$; since $\Delta(\Sigma)^*$ and $\Delta(\Sigma)$ have the same cardinality, we can conclude that $|\overline{\Pi}| \leq \phi(\mathcal{S}^*) = \phi(\mathcal{S})$. ∎

We will give now some examples of skeletons and computations of their final cardinals. The first two examples show that both $2^{\mathrm{cof}(\mathcal{P})} < \phi(\mathcal{S}) = 2^{\phi(\mathcal{P})}$ and $2^{\mathrm{cof}(\mathcal{P})} = \phi(\mathcal{S}) < 2^{\phi(\mathcal{P})}$ are possible. The third example shows that the equality $|\overline{\Gamma}/\Gamma| = \phi(\mathcal{S})$ can actually happen, even if $\overline{\Gamma}/\Gamma$ is free, hence the upper bound in Lemma 9 is the best possible, in general.

Examples: (1) Consider the skeleton $\mathcal{S} = [\mathcal{P} = \omega_1\omega;\ Z_\sigma,\ \sigma \in \mathcal{P}]$, where $Z_\sigma = \mathbb{Z}$ for each σ. Then $\mathrm{cof}(\mathcal{P}) = \aleph_0$ and $\phi(\mathcal{P}) = \aleph_1$; the Hahn product $\mathbb{H}(\mathcal{S})$ coincides with $\Pi_{n\in\omega}G_n$, where $G_n \cong \prod_{\sigma\in\omega_1} Z_\sigma$ for each n. There follows that $\phi(\mathcal{S}) = (\aleph_0^{\aleph_1})^{\aleph_0} = 2^{\aleph_1} = 2^{\phi(\mathcal{P})}$.

(2) Let $\mathcal{S} = [\mathcal{P} = \omega_1^{op}\omega; Z_\sigma, \sigma \in \mathcal{P}]$, where $Z_\sigma = \mathbb{Z}$ for each σ. Then $\mathrm{cof}(\mathcal{P}) = \aleph_0$ and $\phi(\mathcal{P}) = \aleph_1$; $\mathbb{H}(\mathcal{S})$ coincides with $\Pi_{n\in\omega}G_n$, where $G_n \cong \bigoplus_{\sigma\in\omega_1^{op}} Z_\sigma$ for each n. There follows that $\phi(\mathcal{S}) = \aleph_1^{\aleph_0} = 2^{\aleph_0} = 2^{\mathrm{cof}(\mathcal{P})}$.

(3) Let $\mathcal{S} = [\mathcal{P} = \omega_1; Z_\sigma, \sigma \in \mathcal{P}]$, where $Z_\sigma = \mathbb{Z}$ for each σ. Then $\mathbb{H}(\mathcal{S}) = \mathbb{Z}^{\omega_1}$, hence $\phi(\mathcal{S}) = 2^{\aleph_1}$. On the other hand, if we consider the subgroup Γ of $\mathbb{Z}^{\omega_1}$ consisting of the functions of finite range and eventually zero, then Γ is a Specker group over $\mathbb{Z}$ whose closure in $\mathbb{Z}^{\omega_1}$ is the Specker group $\overline{\Gamma}$ consisting of the functions of finite range. By Nöbeling's result [N], $\overline{\Gamma}/\Gamma$ is free and it has cardinality $2^{\aleph_1}$ (see [EM, II.4.9]).

The argument used in the example (3) above is the core of the more general result obtained in Theorem 12 below. We will need the following technical result, which generalizes to arbitrary cardinals, for torsionfree groups, a result in [GH].

LEMMA 10: *Let $\{G_n\}_{n\in\omega}$ be a sequence of non-zero torsionfree abelian groups such that, for every $n \in \omega$, $|G_n| \geq 2^\kappa$ and $\dim G_n/pG_n \geq 2^{\kappa_p}$ (κ and κ_p are infinite cardinals). Then the algebraically compact group $G = \prod_{n\in\omega} G_n / \bigoplus_{n\in\omega} G_n$ has the divisible part of cardinality $\geq 2^\kappa$ and, for each prime p, the invariant of its reduced p-adic component is $\geq 2^{\kappa_p}$.*

Proof: We will use the following: given a sequence $(X_n)_{n\in\omega}$ of sets, each one of infinite cardinal α, and the equivalence relation on the cartesian product $\prod_{n\in\omega} X_n$ defined by setting $(x_n)_{n\in\omega} \approx (y_n)_{n\in\omega}$ if $x_n = y_n$ for almost all n, the quotient set $\prod_{n\in\omega} X_n/\approx$ has cardinality $\alpha^{\aleph_0}$. Let δ be the cardinality of the maximal divisible subgroup of G. Denote by π the canonical epimorphism

$$\pi\colon \prod_{n\in\omega} G_n \to \prod_{n\in\omega} G_n \Big/ \bigoplus_{n\in\omega} G_n;$$

then the set $\pi(\prod_{n\in\omega} n!G_n)$ has cardinality $\geq 2^\kappa$ and it consists of elements divisible by every integer, hence $\delta \geq 2^\kappa$. Recall now that the invariant of the p-adic component of G is the dimension of G/pG as $\mathbb{Z}(p)$-vector space. Since κ_p is infinite, the cardinality of G_n/pG_n coincides with its dimension over $\mathbb{Z}(p)$ and it is $\geq 2^{\kappa_p}$. Thus it is enough to prove the existence in G/pG of 2^{κ_p} distinct elements. For every $n \in \omega$, let $Y_n = \{x_{n\gamma} \in G_n\colon \gamma \in 2^{\kappa_p}\}$ be a set of 2^{κ_p} representatives of a basis of G_n/pG_n; consider the set $Y = \prod_{n\in\omega} Y_n$. Clearly $|\pi(Y)| = 2^{\kappa_p}$; moreover two distinct elements $\pi(y_1)$ and $\pi(y_2)$ are congruent modulo pG if and only if there exists an index n_0 such that the differences of the n-th components $y_1(n) - y_2(n)$ are in pG_n for every $n \geq n_0$, since

$$G/pG \cong \prod_{n\in\omega} G_n \Big/ \left(\prod_{n\in\omega} pG_n + \bigoplus_{n\in\omega} G_n\right).$$

But this is impossible, by our choice of the elements in Y_n, so we are done. ∎

First we will give a realization theorem in the case of countable cofinality, for groups which must be necessarily cotorsion, by Proposition 5.

THEOREM 11: *Let $\mathcal{S} = [\mathcal{P}; \Delta(\Sigma), \Sigma \in \mathcal{P}]$ be a skeleton such that $\mathrm{cof}(\mathcal{P}) = \aleph_0$ and let G be a cotorsion group. Then G is $\mathcal{S}$-realizable in each one of the following cases:*

(1) *$\mathcal{P}$ contains a cofinal subset order isomorphic to the ordinal $\kappa\omega$, where κ is an infinite cardinal, and G has cardinality $\leq 2^\kappa$;*

(2) *$\mathcal{P}$ does not contain a cofinal subset order isomorphic to the ordinal $\kappa\omega$, for any infinite cardinal κ, G has cardinality $\leq 2^{\aleph_0}$ and its reduced p-adic components are trivial for all primes p such that the groups $\Delta(\Sigma)$ are p-divisible for all Σ in a final interval.*

Proof: In all cases we will realize the cotorsion group G as $\overline{\Gamma}/\Gamma$, where Γ satisfies the inclusions (a). Let $G = \prod_p G_p \oplus d(G)$, where $d(G)$ is the divisible part of G and G_p the reduced p-adic component of G. We have an epimorphism

$$A = \prod_p A_p \oplus D \to G \tag{b}$$

where $A_p = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), D_p/G_p)$, with D_p the divisible hull of G_p and D a divisible torsionfree group of the same cardinality as $d(G)$. Each group A_p is torsionfree algebraically compact of cardinality at most $|G_p|^{\aleph_0}$, and it is trivial provided G_p is trivial. Hence the cardinal hypothesis on G holds also for A. If we realize A as $\overline{\Gamma}/\Gamma$, with $\overline{\Gamma}$ and Γ as in (a), then $G \cong \overline{\Gamma}/\Gamma'$, with Γ' containing Γ; then we will be done.

(1) Let $\Gamma = \Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_n < \cdots$ be a cofinal sequence in $\mathcal{P}$ such that each interval $[\Gamma_n, \Gamma_{n+1}]$ contains a subset order isomorphic to κ. Let A be the group in (b), of cardinality $\leq 2^\kappa$. Consider the direct product $\prod_{n\in\omega} \Omega_n$ with the lexicographic ordering, where $\Omega_n = \mathbb{H}_{\Gamma_n \leq \Sigma < \Gamma_{n+1}} \Delta(\Sigma)$. Each group Ω_n contains a subgroup isomorphic to $\mathbb{Z}^\kappa$, hence it contains a subgroup $H_n = (\bigoplus_{\Gamma_n \leq \Sigma < \Gamma_{n+1}} \Delta(\Sigma)) \oplus F_n$, where F_n is a free group of cardinality

2^κ; the existence of such an F_n follows by simple cardinal considerations. The closure of the group $\bigoplus_{n\in\omega} H_n$ in the complete group $\prod_{n\in\omega}\Omega_n = \mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$ coincides with $\prod_{n\in\omega} H_n$. Since $|H_n|$ and $|H_n/pH_n|$ are both $\geq 2^\kappa$, the factor group $\prod_{n\in\omega} H_n/\bigoplus_{n\in\omega} H_n$ has reduced p-adic components whose invariants are at least 2^κ for all primes p, and a divisible part of cardinality at least 2^κ, in view of Lemma 10. Hence the algebraically compact group A is a quotient of this group.

(2) Let A be the group in (b) with cardinality $\leq 2^{\aleph_0}$. If $\mathcal{P}$ does not contain a cofinal subset order isomorphic to $\kappa\omega$ for any infinite cardinal κ, then, for each cofinal sequence $\Gamma = \Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_n < \cdots$ in $\mathcal{P}$, there exist an $n_0 \in \omega$ and Π sufficiently close to 0 such that

$$\mathbb{H}_{\Sigma\geq\Pi}\Delta(\Sigma) = \prod_{n\geq n_0}\Omega_n,$$

where $\Omega_n = \bigoplus_{\Gamma_n\leq\Sigma<\Gamma_{n+1}}\Delta(\Sigma)$. If, for a fixed prime p, the groups $\Delta(\Sigma)$ are p-divisible for all Σ in a final interval, our hypothesis ensures that $A_p = 0$. Otherwise, there is an infinite set of non-p-divisible groups Ω_n. Then, by the results in [GH], A is a quotient of $\prod_{n\in\omega}\Omega_n/\bigoplus_{n\in\omega}\Omega_n$. ∎

We will deal now with the fine realizations in the case of $\mathrm{cof}(\mathcal{P}) > \aleph_0$.

THEOREM 12: *Let $\mathcal{S} = [\mathcal{P};\Delta(\Sigma),\Sigma\in\mathcal{P}]$ be a skeleton such that $\mathrm{cof}(\mathcal{P}) > \aleph_0$ and let G be an abelian group. Then G is $\mathcal{S}$-realizable in each one of the following cases:*

(1) *G has cardinality $\leq 2^{\mathrm{cof}(\mathcal{P})}$;*

(2) *$\mathcal{P}$ contains a cofinal subset order isomorphic to the ordinal $\kappa\mathrm{cof}(\mathcal{P})$, where κ is an infinite cardinal, and G has cardinality $\leq 2^\kappa$;*

(3) *$\mathcal{P}$ contains a cofinal chain $\Gamma = \Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_\sigma < \cdots$ ($\sigma \in \mathrm{cof}(\mathcal{P})$) such that each interval $[\Gamma_\sigma,\Gamma_{\sigma+1}]$ has infinite cardinality κ and G has cardinality $\leq \kappa 2^{\mathrm{cof}(\mathcal{P})}$.*

Proof: (1) Let $\mathcal{J} \cong \mathrm{cof}(\mathcal{P})$ be a well ordered cofinal subset of $\mathcal{P}$. Since each group $\Delta(\Sigma)$ contains a copy $Z(\Sigma)$ of $\mathbb{Z}$, we can consider the subgroup $\mathbb{H}_{\Sigma\in\mathcal{J}} Z(\Sigma)$ of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$, which is clearly isomorphic to $\mathbb{Z}^{\mathrm{cof}(\mathcal{P})}$. Hence each element of $\mathbb{H}_{\Sigma\in\mathcal{J}} Z(\Sigma)$ can be viewed as a function from $\mathcal{J}$ into $\mathbb{Z}$. Let S be the subgroup of $\prod_{\Sigma\in\mathcal{J}} Z(\Sigma)$ consisting of those functions which have finite range and are eventually zero. S is obviously a Specker group on $\mathbb{Z}$. It is easy to see that the completion $\overline{S}$ of S, with respect to the topology induced on S by the order topology on $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$, is the subgroup of $\prod_{\Sigma\in\mathcal{J}} Z(\Sigma)$ consisting of all functions with finite range, which is also a Specker group on $\mathbb{Z}$. By Nöbeling's result [N], $\overline{S} = S \oplus F$, where F is free. By Corollary 4.9 in [EM, II] and its remark, F has cardinality $2^{\mathrm{cof}(\mathcal{P})}$. Consider now the subgroup of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$:

$$\Gamma = \left(\bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)\right) + S$$

endowed with the total ordering induced by that on $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$. The hypothesis that $\mathrm{cof}(\mathcal{P}) > \aleph_0$ ensures that $\bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ is closed in $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$. We will show now that the closure $\overline{\Gamma}$ of Γ in $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$ is the group $X = (\bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)) + \overline{S}$.

It is enough to prove that X is closed. Let $\{x_{\Sigma_\mu}\}_{\mu\in\mathrm{cof}(\mathcal{P})}$ be a Cauchy net of elements in X; we must show that its limit is in X. Without loss of generality, we can assume that

$$x_{\Sigma_\mu} \in \mathbb{H}_{\Sigma<\Sigma_\mu}\Delta(\Sigma) \cap X$$

and

$$x_{\Sigma_\nu}(\Sigma) = x_{\Sigma_\mu}(\Sigma) \quad \text{for every } \Sigma < \Sigma_\mu < \Sigma_\nu.$$

Let $x_{\Sigma_\mu} = d_{\Sigma_\mu} + s_{\Sigma_\mu}$, where $d_{\Sigma_\mu} \in \bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ and $s_{\Sigma_\mu} \in \overline{S}$. Let F_μ be the support of d_{Σ_μ}. Now we define a new element $d'_{\Sigma_\mu} \in \bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ in the following way:

$$d'_{\Sigma_\mu}(\Sigma) = \begin{cases} d_{\Sigma_\mu}(\Sigma)\,(= x_{\Sigma_\mu}(\Sigma)) & \text{if } \Sigma \in F_\mu \setminus \mathcal{J}, \\ x_{\Sigma_\mu}(\Sigma) & \text{if } \Sigma \in F_\mu \bigcap \mathcal{J} \text{ and } x_{\Sigma_\mu}(\Sigma) \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $x_{\Sigma_\mu} = d'_{\Sigma_\mu} + s'_{\Sigma_\mu}$, where $s'_{\Sigma_\mu} \in \overline{S}$ has support disjoint from the support F'_μ of d'_{Σ_μ}; moreover, if $\nu > \mu$, then $F'_\mu \subseteq F'_\nu$. Since $\mathrm{cof}(\mathcal{P}) > \aleph_0$, there follows that $F = \bigcup_{\mu\in\mathrm{cof}(\mathcal{P})} F'_\mu$ is a finite subset of $\mathcal{P}$. By our choice of the elements d'_{Σ_μ} it is clear that the sequence $\{d'_{\Sigma_\mu}\}_{\mu\in\mathrm{cof}(\mathcal{P})}$ is a Cauchy net and it converges to an element $d \in \bigoplus_{\Sigma\in\mathcal{P}} \Delta(\Sigma)$ with support F. This implies also that $\{s'_{\Sigma_\mu}\}_{\mu\in\mathrm{cof}(\mathcal{P})}$ is a Cauchy net, hence it converges to an element $\overline{s} \in \overline{S}$; thus the Cauchy net $\{x_{\Sigma_\mu}\}_{\mu\in\mathrm{cof}(\mathcal{P})}$ converges to $d + \overline{s} \in X$.

Now the conclusion that $\overline{\Gamma}/\Gamma$ is isomorphic to F easily follows from the inclusion $\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)\bigcap\overline{S}\subseteq S$; in fact we have:

$$\begin{aligned}\overline{\Gamma}/\Gamma &= \bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)+\overline{S}/\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)+S\cong\overline{S}/\overline{S}\cap\left(\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)+S\right)\\ &=\overline{S}/\left(\overline{S}\cap\left(\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)\right)\right)+S=\overline{S}/S\cong F.\end{aligned}$$

Hence G is $\mathcal{S}$-realizable, since it is an epimorphic image of F.

(2) There is an ascending cofinal chain in $\mathcal{P}$: $\Gamma=\Gamma_0<\Gamma_1<\Gamma_2<\cdots<\Gamma_\sigma<\cdots$ ($\sigma\in\operatorname{cof}(\mathcal{P})$), such that, for every σ, the interval $[\Gamma_\sigma,\Gamma_{\sigma+1}]$ contains a subset $\mathcal{K}_\sigma$ order isomorphic to κ. Let us consider, in the subgroup $\prod_{\Sigma\in\mathcal{K}_\sigma}\Delta(\Sigma)$ of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$, a free group Ω_σ of rank 2^κ, and the subgroup $\prod_{\sigma\in\operatorname{cof}(\mathcal{P})}\Omega_\sigma$ of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$. Each element of $\prod_{\sigma\in\operatorname{cof}(\mathcal{P})}\Omega_\sigma$ can be viewed as a function from $\operatorname{cof}(\mathcal{P})$ into $A=\bigoplus_{2^\kappa}\mathbb{Z}$. Let T be the subgroup of $\prod_{\sigma\in\operatorname{cof}(\mathcal{P})}\Omega_\sigma$ consisting of those functions which have finite range and are eventually zero. T is obviously a Specker group on A. Consider the topology induced on T by the order topology on $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$; it is easy to see that the completion $\overline{T}$ of T is the subgroup of $\prod_{\sigma\in\operatorname{cof}(\mathcal{P})}\Omega_\sigma$ consisting of all functions with finite range, which is also a Specker group on A. By the Kaup and Kean's version of Nöbeling's result [KK], $\overline{T}=T\oplus E$, where E is a free group with a characteristic A-basis, which has cardinality 2^κ. Consider now the subgroup of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$:

$$\Gamma=\left(\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)\right)+T$$

endowed with the total ordering induced by the order of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$. Arguments similar to those used before show that the closure of Γ in $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$ is

$$\overline{\Gamma}=\left(\bigoplus_{\Sigma\in\mathcal{P}}\Delta(\Sigma)\right)+\overline{T}$$

and that $\overline{\Gamma}/\Gamma\cong E\cong\bigoplus_{2^\kappa}\mathbb{Z}$. G is clearly an epimorphic image of E, hence it is $\mathcal{S}$-realizable.

(3) In this case the proof is just a small modification of the proof in (2): for each interval $[\Gamma_\sigma,\Gamma_{\sigma+1}]$ the subgroup $\prod_{\Gamma_\sigma\leq\Sigma\leq\Gamma_{\sigma+1}}\Delta(\Sigma)$ of $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma)$ contains a subgroup isomorphic to $B=\bigoplus_\kappa\mathbb{Z}$; repeat now the preceding proof with B

instead of A, and consider that the analogous group $\overline{T}/T$ has cardinality $\kappa 2^{\mathrm{cof}(\mathcal{P})}$. ∎

It is easy to show that a cardinal κ satisfies: (a) the hypothesis in point (2) of Theorem 12 provided that κ is a regular cardinal strictly larger than $\mathrm{cof}(\mathcal{P})$ and it is order embeddable in the final interval starting at Π, for every $\Pi < 0$; (b) the hypothesis in point (3) of Theorem 12 provided that the final cardinal $\phi(\mathcal{P})$ of $\mathcal{P}$ is a regular cardinal strictly larger than $\mathrm{cof}(\mathcal{P})$. By means of two examples very similar to those after Lemma 9, we will show now that point (2) in Theorem 12 gives a better result than point (3), or vice versa, depending on the structure of $\mathcal{P}$.

Examples: (1) Let $\mathcal{S} = [\mathcal{P} = \kappa^{op}\omega_1;\ \Delta(\Sigma) = \mathbb{Z}, \Sigma \in \mathcal{P}]$. Then $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma) \cong \prod_{\omega_1}(\bigoplus_\kappa \mathbb{Z})$, hence $\phi(\mathcal{S}) = \kappa^{\aleph_1}$; the largest cardinal satisfying the hypothesis in point (2) of Theorem 12 is clearly $\aleph_1$, hence that theorem guarantees that free groups of rank $2^{\aleph_1}$ are $\mathcal{S}$-realizable. On the other hand, the cardinal κ satisfies the hypothesis in point (3) of Theorem 12, hence free groups of rank $\kappa 2^{\aleph_1}$ are $\mathcal{S}$-realizable: a better result if κ is large.

(2) Let $\mathcal{S} = [\mathcal{P} = \kappa\omega_1; \Delta(\Sigma) = \mathbb{Z}, \Sigma \in \mathcal{P}]$, where $\kappa \geq 2^{\aleph_1}$. Then $\mathbb{H}_{\Sigma\in\mathcal{P}}\Delta(\Sigma) \cong \prod_{\omega_1}(\prod_\kappa \mathbb{Z})$, hence $\phi(\mathcal{S}) = 2^\kappa$; the cardinal κ satisfies the hypothesis in point (2) of Theorem 12, hence that theorem guarantees that free groups of rank 2^κ are $\mathcal{S}$-realizable. On the other hand, the greatest cardinal satisfying the hypothesis in point (3) of Theorem 12 is κ, hence free groups of rank $\kappa 2^{\aleph_1} = \kappa$ are $\mathcal{S}$-realizable: a worse result.

We will conclude with the following characterization of $\mathcal{S}$-realizable groups, under a particular assumption on the cardinal invariants of $\mathcal{P}$.

Corollary 13: *Let $\mathcal{S} = [\mathcal{P}; \Delta(\Sigma), \Sigma \in \mathcal{P}]$ be a skeleton such that $2^{\phi(\mathcal{P})} = 2^{\mathrm{cof}(\mathcal{P})}$. Then an abelian group G is $\mathcal{S}$-realizable if and only if:*

(1) *$|G| \leq 2^{\mathrm{cof}(\mathcal{P})}$ and*

(2) *if $\mathrm{cof}(\mathcal{P}) = \aleph_0$, then G is cotorsion and, in case $\mathcal{P}$ does not contain a cofinal subset order isomorphic to the ordinal $\aleph_0\omega$ and the groups $\Delta(\Sigma)$ are p-divisible in a final interval for some prime p, then the reduced p-adic component of G is trivial.*

Proof: In view of Proposition 7, Lemma 9 and Theorems 11 and 12, we must only prove the necessity in point (2). Let $\Gamma = \Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_n <$

$\cdots$ be a cofinal sequence in $\mathcal{P}$; since $\mathcal{P}$ does not contain a cofinal subset order isomorphic to $\aleph_0\omega$, there exists an index n_0 such that every non-zero element in the final interval starting at Γ_{n_0} has an immediate predecessor (with respect to the ordering in $\mathcal{P}$) and does not contain infinite ascending sequences. Now we want to show that, if the groups $\Delta(\Sigma)$ in the final interval generated by Γ_{n_0} are p-divisible for some prime p, then $\overline{\Gamma}_{n_0}$ is p-divisible; this would imply that $G = \overline{\Gamma}/\Gamma \cong \overline{\Gamma}_{n_0}/\Gamma_{n_0}$ is p-divisible, i.e. the reduced p-adic component of G is trivial. Since $\overline{\Gamma}_{n_0}$ embeds as a pure subgroup into $\prod_{n>n_0} \Gamma_{n_0}/\Gamma_n$, it is enough to prove that each quotient group Γ_{n_0}/Γ_n if p-divisible. But this is clear, since Γ_{n_0}/Γ_n is a union of a well ordered ascending chain (with respect to the inclusion) of groups such that the quotient of each group of the chain modulo its immediate predecessor (with respect to the inclusion), if such a predecessor exists, is p-divisible. (Note that the groups indexed by limit ordinals in the well ordered chain are non-principal convex subgroups, and hence they are not in $\mathcal{P}$.)
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